For Online Publication

Equilibrium Data Mining

Jérôme Dugast* Thierry Foucault†

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1. Searching predictors with recall

In the baseline model, we assume that speculators's search for predictors is without recall: When they decide to stop searching, they must trade on the predictor they just discovered. An alternative assumption is that when they stop searching, speculators can use the best of all predictors found until they stop (i.e., they can "recall" predictors found in the past). In this section, we show that this extra flexibility does not affect the equilibrium stopping rule, θ^*.

Step 1. To do so, we first derive the continuation value of a speculator who follows an arbitrary stopping rule $\hat{\theta}$ when other speculators follow the stopping rule θ^*. A key difference with the baseline case is that this continuation value in a given exploration round depends on the best predictor she found until this round. Let denote this best predictor by θ^{min}. Observe that it must be the case that $\theta^{min} \geq \hat{\theta}$ (if the inequality was not satisfied the speculator would have stopped searching in a previous round; a

*Université Paris Dauphine - PSL. Tel: (+33) 01 44 05 40 41 ; E-mail: jerome.dugast@dauphine.psl.eu
†HEC, Paris and CEPR. Tel: (33) 1 39 67 95 69; E-mail: foucault@hec.fr

contradiction). The expected utility of launching a new exploration for the speculator is:

$$J(\theta^{min},\widehat{\theta},\theta^*) = \exp(\rho c)\left[\alpha\left(\int_{\underline{\theta}}^{\widehat{\theta}} g(\theta',\theta^*)\phi(\theta')d\theta' + \int_{\widehat{\theta}}^{\theta^{min}} J(\theta',\widehat{\theta},\theta^*)\phi(\theta')d\theta'\right) + \left(1-\alpha\int_{\underline{\theta}}^{\theta^{min}}\phi(\theta')d\theta'\right)J(\theta,\widehat{\theta},\theta^*)\right] \tag{1}$$

Differentiating both sides of eq.(1) with respect to θ^{min} implies that:

$$\frac{\partial J}{\partial\theta^{min}}\left(1-\exp(\rho c)\left(1-\alpha\int_{\underline{\theta}}^{\theta^{min}}\phi(\theta')d\theta'\right)\right) = \alpha\exp(\rho c)\left[J(\theta^{min},\widehat{\theta},\theta^*)\phi(\theta) - \phi(\theta)J(\theta^{min},\widehat{\theta},\theta^*)\right] = 0 \tag{2}$$

Thus, $\frac{\partial J(\theta^{min},\widehat{\theta},\theta^*)}{\partial\theta^{min}} = 0$. Hence, $J(\theta^{min},\widehat{\theta},\theta^*)$ does not depend on θ^{min} for $\theta^{min} > \widehat{\theta}$. This means that for $\theta^{min} > \theta' > \widehat{\theta}$, $J(\theta',\widehat{\theta},\theta^*) = J(\theta^{min},\widehat{\theta},\theta^*)$. Thus, eq.(1) implies:

$$J(\theta^{min},\widehat{\theta},\theta^*) = J(\widehat{\theta},\theta^*) = \exp(\rho c)\left[\alpha\int_{\underline{\theta}}^{\widehat{\theta}} g(\theta',\theta^*)\phi(\theta')d\theta' + \left(1-\alpha\int_{\underline{\theta}}^{\widehat{\theta}}\phi(\theta')d\theta'\right)J(\widehat{\theta},\theta^*)\right], \tag{3}$$

Hence, the continuation value of the speculator for any $\theta^{min} \geq \hat{\theta}$ is the same as in the baseline version of the model. Intuitively, the reason is that there is no limit to the number of explorations that a speculator can perform. Hence, the possibility for a speculator to "store" predictors has no value since the speculator keeps searching as long as the quality of the predictors she found is above $\widehat{\theta}$.

Step 2. Now, suppose that there exists a solution, $\widehat{\theta}$, to the following the indifference condition

$$g(\widehat{\theta},\theta^*) = J(\widehat{\theta},\widehat{\theta},\theta^*). \tag{4}$$

We show below that $\widehat{\theta}$ is the optimal stopping rule of the speculator when other speculators' stopping rule θ^*. According to the one-shot deviation principle, a necessary and sufficient condition for $\widehat{\theta}$ to be optimal is that a one-shot deviation from this policy is not optimal. First, we show that it is always optimal to keep searching when $\theta > \widehat{\theta}$ (thus a one shot deviation that consists in stopping is not optimal), because

$$g(\theta,\theta^*) < g(\widehat{\theta},\theta^*) = J(\widehat{\theta},\widehat{\theta},\theta^*) = J(\theta,\widehat{\theta},\theta^*), \tag{5}$$

where the first inequality follows from the fact that $g(\theta,\theta^*)$ decreases with θ, the second equality follows from eq.(4), and the last equality follows from the fact that $J(\theta,\widehat{\theta},\theta^*)$ does not depend on θ for $\theta \geq \widehat{\theta}$ (see Step 1).

Second, consider a case where the speculator's obtains a predictor of type $\theta \in [\underline{\theta},\widehat{\theta}]$. If stopping when $\theta < \widehat{\theta}$ is the optimal policy then a one shot deviation from this policy is not optimal. We now show that this is the case. The expected continuation value of the speculator with such a one shot deviation is:

$$X(\theta,\theta^*) = \exp(\rho c)\left[\alpha\int_{\underline{\theta}}^{\theta} g(\theta',\theta^*)\phi(\theta')d\theta' + \left(1-\alpha\int_{\underline{\theta}}^{\theta}\phi(\theta')d\theta'\right)g(\theta,\theta^*)\right]. \tag{6}$$

Indeed, either the speculator finds an even better predictor in the next exploration round (first term in the R.H.S of eq.(6)) or she does not and then her policy will command to stop at the next round with the best predictor found so far, i.e., the predictor of type θ (second term in eq.(6)). Notice that $X(\widehat{\theta},\theta^*) = J(\widehat{\theta},\widehat{\theta},\theta^*) = g(\widehat{\theta},\theta^*)$ which shows that a speculator with $\theta = \widehat{\theta}$ is indifferent between deviating or not. Now, we show that:

$$\forall\,\theta < \widehat{\theta},\ g(\theta,\theta^*) > X(\theta,\theta^*). \tag{7}$$

As $g(\theta,\theta^*) < 0$, using the expression for $X(\theta,\theta^*)$, this inequality is equivalent to

$$\forall\,\theta < \widehat{\theta},\ \exp(-\rho c) < Y(\theta,\theta^*), \tag{8}$$

where $Y(\theta,\theta^*) = \alpha\int_{\underline{\theta}}^{\theta}\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}\phi(\theta')d\theta' + 1 - \alpha\int_{\underline{\theta}}^{\theta}\phi(\theta')d\theta'$. Now:

$$\frac{\partial Y}{\partial\theta} = \int_{\underline{\theta}}^{\theta}\frac{\partial}{\partial\theta}\left(\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}\right)\phi(\theta')d\theta'. \tag{9}$$

The expected utility $g(\theta,\theta^*)$ is negative, and is decreasing in θ. Therefore $\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}$ is positive and also decreasing in θ. Thus, $\frac{\partial Y}{\partial \theta} < 0$ and therefore:

$$\forall\, \theta < \theta^*,\ \exp(-\rho c) < Y(\theta, \theta^*), \tag{10}$$

or equivalently $\forall\, \theta < \widehat{\theta},\ g(\theta, \theta^*) > X(\theta, \theta^*)$. Thus, $\widehat{\theta}$ is the optimal stopping rule for the speculator. Moreover, in a symmetric equilibrium, it must be the case that $\widehat{\theta}(\theta^*) = \theta^*$. This means that:

$$g(\theta^*, \theta^*) = J(\theta^*, \theta^*, \theta^*). \tag{11}$$

As the continuation value $J(\theta^*, \theta^*, \theta^*)$ is identical to that in the baseline model (see Step 1), the stopping rule θ^* is the same. Thus, the results are unchanged.

2. Supplement to the proof of Proposition 4

Remember from eq.(50) in the text that:

$$r(\theta, \theta^*) = \frac{g(\theta, \theta^*)}{g(\theta^*, \theta^*)} = \left(\frac{\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)}{\rho^2\sigma^2\nu^2\tau(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)} \right)^{\frac{1}{2}}, \tag{12}$$

Thus, using the fact $\tau(\theta) = \cot(\theta)^2$ and $\bar{\tau}(\theta^*; \underline{\theta}, \alpha) = \mathsf{E}\left[\cot^2(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]$, we deduce that:

$$\begin{aligned} \frac{\partial r(\theta, \theta^*)}{\partial \underline{\theta}} = & \frac{\partial\, \mathsf{E}\left[\cot^2(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]}{\partial \underline{\theta}} \mathsf{E}\left[\cot^2(\theta') \middle| \, \underline{\theta} \leq \theta' \leq \theta^*\right] \\ & \times \frac{\rho^2\sigma^2\nu^2(\cot^2(\theta) - \cot^2(\theta^*))}{\left\{\rho^2\sigma^2\nu^2 \cot^2(\theta) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]^2\right\}^{\frac{3}{2}}} \\ & \times \frac{1}{\left\{\rho^2\sigma^2\nu^2 \cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]^2\right\}^{\frac{1}{2}}}. \end{aligned} \tag{13}$$

Now, using the fact that (i) $\tan(\theta) = 1/\cot(\theta)$, (ii) $\cot(\theta)$ ($\tan(\theta)$) decreases (increases with θ) and (iii) for these reasons, $1 - \cot^2(\theta^*)\tan^2(\theta) > 0$, we observe that the term in the second line of eq.(13) can be bounded below as follows:

$$\begin{aligned}
&\frac{\rho^2\sigma^2\nu^2(\cot^2(\theta)-\cot^2(\theta^*))}{\left\{\rho^2\sigma^2\nu^2\cot^2(\theta)+\rho^2\sigma^2\nu^2+\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]^2\right\}^{\frac{3}{2}}}\\
&=\frac{1}{\cot(\theta)}\times\frac{\rho^2\sigma^2\nu^2(1-\cot^2(\theta^*)\tan^2(\theta))}{\left\{\rho^2\sigma^2\nu^2+\rho^2\sigma^2\nu^2\tan^2(\theta)+\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]^2\tan^2(\theta)\right\}^{\frac{3}{2}}}\\
&>\frac{1}{\cot(\underline{\theta})}\times\frac{\rho^2\sigma^2\nu^2(1-\cot^2(\theta^*)\tan^2(\theta))}{\left\{\rho^2\sigma^2\nu^2+\rho^2\sigma^2\nu^2\tan^2(\theta^*)+\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]^2\tan^2(\theta^*)\right\}^{\frac{3}{2}}}>0.
\end{aligned}\tag{14}$$

Thus:

$$\frac{\int_{\underline{\theta}}^{\theta^*}\frac{\partial r}{\partial\underline{\theta}}\phi(\theta)d\theta}{\phi(\underline{\theta})}<-\left(K(\underline{\theta},\theta^*)\frac{\cot^2(\underline{\theta})-\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]}{\cot(\underline{\theta})}\left(1-\cot^2(\theta^*)\,\mathsf{E}\left[\tan^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]\right)\right)\tag{15}$$

where

$$\begin{aligned}
K(\underline{\theta},\theta^*)=\;&\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]\\
&\times\frac{\rho^2\sigma^2\nu^2}{\left\{\rho^2\sigma^2\nu^2+\rho^2\sigma^2\nu^2\tan^2(\theta^*)+\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]^2\tan^2(\theta^*)\right\}^{\frac{3}{2}}}\\
&\times\frac{1}{\left\{\rho^2\sigma^2\nu^2\cot^2(\theta^*)+\rho^2\sigma^2\nu^2+\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]^2\right\}^{\frac{1}{2}}}>0.
\end{aligned}\tag{16}$$

By Assumption A.1, $\bar{\tau}(\theta^*;\underline{\theta},\alpha)=\mathsf{E}\left[\cot^2(\theta')\middle|\,0\leq\theta'\leq\theta^*\right]$ exists (does not diverge) for all θ^*. Moreover, $\theta^*>\underline{\theta}$ for all values of $\underline{\theta}\geq 0$ (Proposition 2). Thus, $\mathsf{E}\left[\cot^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]$ is bounded, $K(\underline{\theta},\theta^*)$ admits a lower bound that is strictly positive, and $\cot^2(\theta^*)\,\mathsf{E}\left[\tan^2(\theta')\middle|\,\underline{\theta}\leq\theta'\leq\theta^*\right]$ admits an upper bound that is strictly lower than one. Moreover $\cot(\underline{\theta})$ becomes infinite when $\underline{\theta}$ goes to zero. We deduce from eq.(15) that $\frac{\int_{\underline{\theta}}^{\theta^*}\frac{\partial r}{\partial\underline{\theta}}\phi(\theta)d\theta}{\phi(\underline{\theta})}$ goes to $-\infty$ when $\underline{\theta}$ goes to zero.

3. Special Cases

In this section we consider two particular specifications for the probability distribution of the predictors, $\phi(.)$. As explained below, in these cases, we can easily compute numerically the equilibrium stopping rule (θ^*) and various variables of interest that depends on the equilibrium stopping rule (e.g., the mean and the variance of the distribution of the quality of predictors across speculators). For this reason, we use these specifications to illustrate numerically the main results of the paper.

3.1 Case 1: $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$.

In this case, the cumulative probability distribution for the type of predictors is $\Phi(\theta) = \sin^3(\theta)$. Figure 1 plots $\phi(\theta)$ in this case.

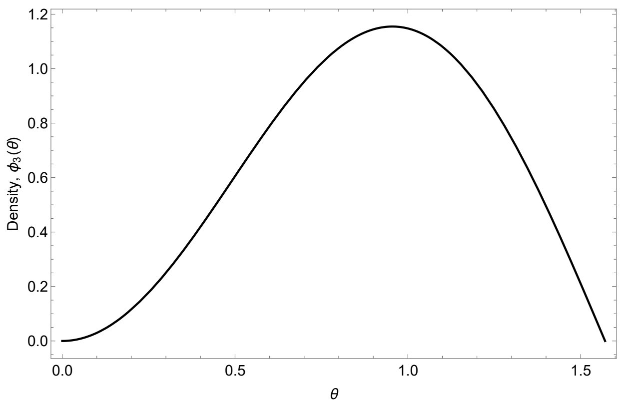

Figure 1: This graph represents the distribution function $\phi_(\theta) = 3\cos(\theta)\sin^2(\theta)$ on the interval $[0, \pi/2]$.

We obtain the following result regarding the two first moments of the distribution of the quality of predictor chosen by speculators in equilibrium (i.e., the distribution of $\tau(\theta)$ for the types of predictors used in equilibrium). Remember that: $\bar{\tau}(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\tau(\theta')\middle|\, \underline{\theta} \le \theta' \le \theta^*\right]$ and $m_2(\theta^*, \underline{\theta}) \equiv \mathsf{E}\left[\tau^2(\theta')\middle|\, \underline{\theta} \le \theta' \le \theta^*\right]$.

Lemma 1. *When $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$, we have:*

$$\bar{\tau}(\theta^*, \underline{\theta}, \alpha) = \frac{3[\sin(\theta^*) - \sin(\underline{\theta})] - [\sin^3(\theta^*) - \sin^3(\underline{\theta})]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \tag{17}$$

$$m_2(\theta^*, \underline{\theta}) = \frac{\left[3\left(\frac{1}{\sin(\underline{\theta})} - \frac{1}{\sin(\theta^*)}\right) - 6(\sin(\theta^*) - \sin(\underline{\theta})) + \sin^3(\theta^*) - \sin^3(\underline{\theta})\right]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \tag{18}$$

Remember that the equilibrium stopping rule θ^* solves: $exp(-\rho c) = F(\theta^*)$. The next lemma provides the expression for $F(\theta^*)$ in Case 1.

Lemma 2. *When* $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$, *we have:*

$$F(\theta^*) = 1 - (\sin^3(\theta^*) - \sin^3(\underline{\theta})) \tag{19}$$

$$+ \left(\rho^2\sigma^2\nu^2\cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2\right)^{\frac{1}{2}} [\delta(\theta^*) - \delta(\underline{\theta})] \tag{20}$$

where $\delta(\theta) = \frac{\left(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2 \sin^2(\theta)\right)^{\frac{1}{2}}}{\bar{\tau}(\theta^*, \underline{\theta}, \alpha)^4} \left(\bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2 \sin^2(\theta) - 2\rho^2\sigma^2\nu^2\right)$ *and* $\bar{\tau}(\theta^*, \underline{\theta}, \alpha)$ *is given in Lemma 1.*

This result, combined with the expressions for the first and the second moments in Lemma 1 enable us to compute, for instance, the first and the second moment of speculators' expected trading profit in Section 5.3. ($\mathsf{E}[\bar{\pi}(\theta)]$ and $\mathsf{Var}[\bar{\pi}(\theta)]$).

3.2 Case 2: $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$.

In this case, the cumulative probability distribution for the type of predictors is: $\Phi(\theta) = \sin^5(\theta)$. Figure 2 plots $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$.

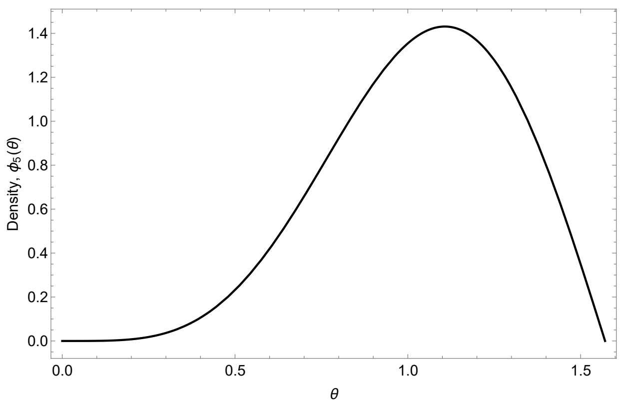

Figure 2: This graph represents the distribution function $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$ on the interval $[0, \pi/2]$.

Lemma 3. *When* $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$,

$$\bar{\tau}(\theta^*, \underline{\theta}, \alpha) = \frac{\frac{5}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] - [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})}. \tag{21}$$

$$m_2(\theta^*, \underline{\theta}) = \frac{5[\sin(\theta^*) - \sin(\underline{\theta})] - \frac{10}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] + [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \tag{22}$$

As in Case 1, the next lemma provides the expression for $F(\theta^*)$.

Lemma 4. *When $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$, we have:*

$$F(\theta^*) = 1 - (\sin^5(\theta^*) - \sin^5(\underline{\theta})) \tag{23}$$
$$+ \left(\rho^2\sigma^2\nu^2\cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2\right)^{\frac{1}{2}} [\delta(\theta^*) - \delta(\underline{\theta})] \tag{24}$$

with $\delta(\theta) = \frac{\left(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta)\right)^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6}\left[\frac{8}{3}\rho^4\sigma^4\nu^4 - \frac{4}{3}\rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta)\right]$ *and* $\bar{\tau}(\theta^*, \underline{\theta}, \alpha)$ *is given in Lemma 3*

As in case 1, the expression for the second moment allows us to compute the volatility of speculators expected profits, $\mathsf{Var}[\pi(\theta)]$.

3.3 Proofs

Proof of Lemma 1. As $\tau(\theta) = \cot^2(\theta)$ and $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$ in Case 1, we have:

$$\bar{\tau}(\theta^*, \underline{\theta}, \alpha) = \mathsf{E}\left[\tau(\theta') | \underline{\theta} \le \theta' \le \theta^*\right] \tag{25}$$
$$= \mathsf{E}\left[\cot^2(\theta')\middle| \underline{\theta} \le \theta' \le \theta^*\right] \tag{26}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 3\cos^3(\theta)d\theta \tag{27}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 3\cos(\theta)(1-\sin^2(\theta))d\theta \tag{28}$$
$$= \frac{3[\sin(\theta^*) - \sin(\underline{\theta})] - [\sin^3(\theta^*) - \sin^3(\underline{\theta})]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})}. \tag{29}$$

Moreover:

$$m_2(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\cot^4(\theta')\middle| \underline{\theta} \le \theta' \le \theta^*\right] \tag{30}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 3\frac{\cos^5(\theta)}{\sin^2(\theta)}d\theta \tag{31}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 3\cos(\theta)\left(\frac{1}{\sin^2(\theta)} - 2 + \sin^2(\theta)\right)d\theta \tag{32}$$
$$= \frac{\left[3\left(\frac{1}{\sin(\underline{\theta})} - \frac{1}{\sin(\theta^*)}\right) - 6(\sin(\theta^*) - \sin(\underline{\theta})) + \sin^3(\theta^*) - \sin^3(\underline{\theta})\right]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \tag{33}$$

Proof of Lemma 2. Remember that (see eq.(19) in the text) that:

$$F(\theta^*) \equiv \alpha \int_{\underline{\theta}}^{\theta^*} r(\theta, \theta^*)\phi(\theta)d\theta + (1 - \Lambda(\theta^*; \underline{\theta}, \alpha)), \quad \text{for } \theta^* \in \left[\underline{\theta}, \frac{\pi}{2}\right], \tag{34}$$

with

$$r(\theta, \theta^*) = \frac{g(\theta, \theta^*)}{g(\theta^*, \theta^*)} = \left(\frac{\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)}{\rho^2\sigma^2\nu^2\tau(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)}\right)^{\frac{1}{2}}, \tag{35}$$

where the expression for $r(\theta, \theta^*)$ follows from eq.(50) in the appendix of the paper.

In Case 1, we have: $(1 - \Lambda(\theta^*; \underline{\theta}, \alpha)) = 1 - \alpha(\sin^3(\theta^*) - \sin^3(\underline{\theta}))$. Moreover,

$$\alpha \int_{\underline{\theta}}^{\theta^*} r(\theta, \theta^*)\phi(\theta)d\theta =$$

$$\alpha \left(\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\right)^{\frac{1}{2}} \int_{\underline{\theta}}^{\theta^*} \frac{3\cos(\theta)\sin^2(\theta)}{(\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha))^{\frac{1}{2}}} d\theta. \tag{36}$$

Now:

$$\int_{\underline{\theta}}^{\theta^*} \frac{3\cos(\theta)\sin^2(\theta)}{(\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha))^{\frac{1}{2}}} d\theta \tag{37}$$

$$= \int_{\underline{\theta}}^{\theta^*} \frac{3\cos(\theta)\sin^3(\theta)}{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}} d\theta \tag{38}$$

$$= 3\int_{\underline{\theta}}^{\theta^*} \frac{\cos(\theta)\sin(\theta)}{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}} \sin^2(\theta)d\theta \tag{39}$$

$$= 3\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)} \sin^2(\theta)\right]_{\underline{\theta}}^{\theta^*} \tag{40}$$

$$- 2\int_{\underline{\theta}}^{\theta^*} \frac{3}{2} \frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)} 2\cos(\theta)\sin(\theta)d\theta \tag{41}$$

$$= 3\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)} \sin^2(\theta)\right]_{\underline{\theta}}^{\theta^*} \tag{42}$$

$$- 2\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{3}{2}}}{\bar{\tau}^4(\theta^*; \underline{\theta}, \alpha)}\right]_{\underline{\theta}}^{\theta^*} \tag{43}$$

$$= \left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^4(\theta^*; \underline{\theta}, \alpha)} \left(\bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\sin^2(\theta) - 2\rho^2\sigma^2\nu^2\right)\right]_{\underline{\theta}}^{\theta^*} \tag{44}$$

Thus, we deduce from eq.(36) that

$$F(\theta^*) = 1 - \alpha(\sin^3(\theta^*) - \sin^3(\underline{\theta})) \tag{45}$$

$$+ \left(\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)\right)^{\frac{1}{2}} [\delta(\theta^*) - \delta(\underline{\theta})] \tag{46}$$

with $\delta(\theta) = \frac{\left(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2 \sin^2(\theta)\right)^{\frac{1}{2}}}{\bar{\tau}(\theta^*, \underline{\theta}, \alpha)^4} (\bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2 \sin^2(\theta) - 2\rho^2\sigma^2\nu^2)$.

Proof of Lemma 3. As by definition $\tau(\theta) = cot^2(\theta)$ and $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$ in Case 2, we have:

$$\begin{aligned} \bar{\tau}(\theta^*, \underline{\theta}, \alpha) &\equiv \mathsf{E}\left[\cot^2(\theta')\middle| \underline{\theta} \le \theta' \le \theta^*\right] \\ &= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 5\cos^3(\theta)\sin^2(\theta) d\theta \\ &= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 5\cos(\theta)(\sin^2(\theta) - \sin^4(\theta)) d\theta \\ &= \frac{\frac{5}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] - [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \end{aligned} \tag{47}$$

and

$$\begin{aligned} m_2(\theta^*, \underline{\theta}) &\equiv \mathsf{E}\left[\cot^4(\theta')\middle| \underline{\theta} \le \theta' \le \theta^*\right] \\ &= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 5\cos^5(\theta) d\theta \\ &= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 5\cos(\theta)(1 - 2\sin^2(\theta) + \sin^4(\theta)) d\theta \\ &= \frac{5[\sin(\theta^*) - \sin(\underline{\theta})] - \frac{10}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] + [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \end{aligned} \tag{48}$$

Proof of Lemma 4. The expression for $F(\theta^*)$ for any $\phi(.)$ is given in eq.(34). In Case 2, we have $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$, so that: $(1 - \Lambda(\theta^*; \underline{\theta}, \alpha)) = 1 - \alpha(\sin^5(\theta^*) - \sin^5(\underline{\theta}))$.

Moreover,

$$\alpha \int_{\underline{\theta}}^{\theta^*} r(\theta,\theta^*)\phi(\theta) d\theta =$$
$$\alpha\left(\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\right)^{\frac{1}{2}} \int_{\underline{\theta}}^{\theta^*} \frac{5\cos(\theta)\sin^4(\theta)}{(\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha))^{\frac{1}{2}}} d\theta. \tag{49}$$

Now:

$$\int_{\underline{\theta}}^{\theta^*} \frac{5\cos(\theta)\sin^4(\theta)}{(\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha))^{\frac{1}{2}}} d\theta \tag{50}$$
$$= \int_{\underline{\theta}}^{\theta^*} \frac{5\cos(\theta)\sin^5(\theta)}{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}} d\theta \tag{51}$$
$$= 5\int_{\underline{\theta}}^{\theta^*} \frac{\cos(\theta)\sin(\theta)}{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}} \sin^4(\theta) d\theta \tag{52}$$
$$= 5\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)} \sin^4(\theta)\right]_{\underline{\theta}}^{\theta^*} \tag{53}$$
$$- 5\int_{\underline{\theta}}^{\theta^*} \frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)} 4\cos(\theta)\sin^3(\theta) d\theta \tag{54}$$
$$= 5\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)} \sin^4(\theta)\right]_{\underline{\theta}}^{\theta^*} \tag{55}$$
$$- \frac{20}{3}\int_{\underline{\theta}}^{\theta^*} \frac{3}{2}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)} 2\cos(\theta)\sin(\theta)\sin^2(\theta) d\theta \tag{56}$$

Then,

$$\int_{\underline{\theta}}^{\theta^*} \frac{3}{2}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)} 2\cos(\theta)\sin(\theta)\sin^2(\theta) d\theta \tag{57}$$
$$= \left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{3}{2}}}{\bar{\tau}^4(\theta^*;\underline{\theta},\alpha)} \sin^2(\theta)\right]_{\underline{\theta}}^{\theta^*} \tag{58}$$
$$- \int_{\underline{\theta}}^{\theta^*} \frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{3}{2}}}{\bar{\tau}^4(\theta^*;\underline{\theta},\alpha)} 2\cos(\theta)\sin(\theta) d\theta \tag{59}$$

and

$$\int_{\underline{\theta}}^{\theta^*} \frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{3}{2}}}{\bar{\tau}^4(\theta^*;\underline{\theta},\alpha)} 2\cos(\theta)\sin(\theta)d\theta \tag{60}$$

$$= \frac{2}{5}\left[\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)\sin^2(\theta))^{\frac{5}{2}}}{\bar{\tau}^6(\theta^*;\underline{\theta},\alpha)}\right]_{\underline{\theta}}^{\theta^*} \tag{61}$$

Thus,

$$F(\theta^*) = 1 - (\sin^5(\theta^*) - \sin^5(\underline{\theta})) \tag{62}$$

$$+ \left(\rho^2\sigma^2\nu^2\cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*,\underline{\theta})\right)^{\frac{1}{2}}[\delta(\theta^*) - \delta(\underline{\theta})] \tag{63}$$

with

$$\delta(\theta) = 5\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta})^2}\sin^4(\theta) \tag{64}$$

$$-\frac{20}{3}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta))^{\frac{3}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^4}\sin^2(\theta) \tag{65}$$

$$+\frac{8}{3}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta))^{\frac{5}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6} \tag{66}$$

$$= \frac{20}{3}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6} \tag{67}$$

$$\times\Big[\frac{3}{4}\bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta) - \rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) - \bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta) \tag{68}$$

$$+\frac{2}{5}\left(\rho^4\sigma^4\nu^4 + 2\rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta)\right)\Big] \tag{69}$$

$$= \frac{20}{3}\frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta})^2\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6}\left[\frac{2}{5}\rho^4\sigma^4\nu^4 - \frac{1}{5}\rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) + \frac{3}{20}\bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta)\right] \tag{70}$$

$$= \frac{(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta))^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6}\left[\frac{8}{3}\rho^4\sigma^4\nu^4 - \frac{4}{3}\rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta)\right] \tag{71}$$