Online Appendix
Equilibrium Data Mining
NOT FOR PUBLICATION

January 3, 2022

Contents

Part I

Additional Proofs

A. Supplement to the proof of Proposition 4

Remember from eq.(52) in the text that:

$$r(\theta,\theta^*) = \frac{g(\theta,\theta^*)}{g(\theta^*,\theta^*)} = \left(\frac{\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)}{\rho^2\sigma^2\nu^2\tau(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*;\underline{\theta},\alpha)}\right)^{\frac{1}{2}}, \tag{I.A.1}$$

Thus, using the fact $\tau(\theta) = \cot(\theta)^2$ and $\bar{\tau}(\theta^*;\underline{\theta},\alpha) = \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]$, we deduce that:

$$\begin{aligned}
\frac{\partial r(\theta,\theta^*)}{\partial \underline{\theta}} = {} & \frac{\partial\, \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]}{\partial \underline{\theta}} \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right] \\
& \times \frac{\rho^2\sigma^2\nu^2(\cot^2(\theta) - \cot^2(\theta^*))}{\left\{\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\right\}^{\frac{3}{2}}} \\
& \times \frac{1}{\left\{\rho^2\sigma^2\nu^2\cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\right\}^{\frac{1}{2}}}.
\end{aligned} \tag{I.A.2}$$

Now, using the fact that (i) $\tan(\theta) = 1/\cot(\theta)$, (ii) $\cot(\theta)$ ($\tan(\theta)$) decreases (increases with θ) and (iii) for these reasons, $1 - \cot^2(\theta^*)\tan^2(\theta) > 0$, we observe that the term in the second line of eq.(I.A.2) can be bounded below as follows:

$$\begin{aligned}
& \frac{\rho^2\sigma^2\nu^2(\cot^2(\theta) - \cot^2(\theta^*))}{\left\{\rho^2\sigma^2\nu^2\cot^2(\theta) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\right\}^{\frac{3}{2}}} \\
= {} & \frac{1}{\cot(\theta)} \times \frac{\rho^2\sigma^2\nu^2(1 - \cot^2(\theta^*)\tan^2(\theta))}{\left\{\rho^2\sigma^2\nu^2 + \rho^2\sigma^2\nu^2\tan^2(\theta) + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\tan^2(\theta)\right\}^{\frac{3}{2}}} \\
> {} & \frac{1}{\cot(\underline{\theta})} \times \frac{\rho^2\sigma^2\nu^2(1 - \cot^2(\theta^*)\tan^2(\theta))}{\left\{\rho^2\sigma^2\nu^2 + \rho^2\sigma^2\nu^2\tan^2(\theta^*) + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\tan^2(\theta^*)\right\}^{\frac{3}{2}}} > 0.
\end{aligned} \tag{I.A.3}$$

Thus:

$$\frac{\int_{\underline{\theta}}^{\theta^*} \frac{\partial r}{\partial \underline{\theta}} \phi(\theta) d\theta}{\phi(\underline{\theta})} < -\left(K(\underline{\theta}, \theta^*) \frac{\cot^2(\underline{\theta}) - \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]}{\cot(\underline{\theta})} \left(1 - \cot^2(\theta^*)\, \mathsf{E}\left[\tan^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]\right)\right) \tag{I.A.4}$$

where

$$\begin{aligned} K(\underline{\theta}, \theta^*) = {} & \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right] \\ & \times \frac{\rho^2\sigma^2\nu^2}{\left\{\rho^2\sigma^2\nu^2 + \rho^2\sigma^2\nu^2 \tan^2(\theta^*) + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2 \tan^2(\theta^*)\right\}^{\frac{3}{2}}} \qquad \text{(I.A.5)} \\ & \times \frac{1}{\left\{\rho^2\sigma^2\nu^2 \cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]^2\right\}^{\frac{1}{2}}} > 0. \end{aligned}$$

By Assumption A.1, $\bar{\tau}(\theta^*; \underline{\theta}, \alpha) = \mathsf{E}\left[\cot^2(\theta') \middle| 0 \le \theta' \le \theta^*\right]$ exists (does not diverge) for all θ^*. Moreover, $\theta^* > \underline{\theta}$ for all values of $\underline{\theta} \ge 0$ (Proposition 2). Thus, $\mathsf{E}\left[\cot^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]$ is bounded, $K(\underline{\theta}, \theta^*)$ admits a lower bound that is strictly positive, and $\cot^2(\theta^*)\, \mathsf{E}\left[\tan^2(\theta') \middle| \underline{\theta} \le \theta' \le \theta^*\right]$ admits an upper bound that is strictly lower than one. Moreover $\cot(\underline{\theta})$ becomes infinite when $\underline{\theta}$ goes to zero. We deduce from eq.(I.A.4) that $\frac{\int_{\underline{\theta}}^{\theta^*} \frac{\partial r}{\partial \underline{\theta}} \phi(\theta) d\theta}{\phi(\underline{\theta})}$ goes to $-\infty$ when $\underline{\theta}$ goes to zero.

B. Proof of Proposition 6

As explained in the text, when the mass of speculators is μ, we can proceed as in Section 4 to derive the following equilibrium condition for θ^*:

$$F(\theta^*, \mu) = \exp(-\rho c), \tag{I.B.1}$$

where

$$F(\theta^*, \mu) \equiv \alpha \int_{\underline{\theta}}^{\theta^*} r(\theta, \theta^*, \mu) \phi(\theta) d\theta + \left(1 - \Lambda(\theta^*; \underline{\theta}, \alpha)\right), \quad \text{for } \theta^* \in \left[\underline{\theta}, \frac{\pi}{2}\right], \tag{I.B.2}$$

with

$$r(\theta, \theta^*, \mu)) \equiv \frac{g(\theta, \theta^*)}{g(\theta^*, \theta^*)} = \left(\frac{\tau(\theta^*)\tau_\omega + \mathcal{I}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)}{\tau(\theta)\tau_\omega + \mathcal{I}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)}\right)^{\frac{1}{2}}, \tag{I.B.3}$$

and

$$\mathcal{I}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha) = \tau_\omega + \frac{\bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)^2 \tau_\omega^2}{\rho^2\nu^2}. \tag{I.B.4}$$

Note that this is the same equilibrium condition as that obtained when $\mu = 1$. The only difference is that, in the expression for $\mathcal{I}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)$, $\bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)$ (defined in eq.(30) in the text) replaces $\bar{\tau}(\theta^*; \underline{\theta}, \alpha) = \bar{\tau}(\theta^*; 1, \underline{\theta}, \bar{\theta}, \alpha)$.

Substituting $\mathcal{I}(\theta^*(\mu); \underline{\theta}, \alpha)$ by its expression in eq.(I.B.4) into eq.(I.B.3), we obtain:

$$r(\theta, \theta^*, \mu) = \left(\frac{\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)^2}{\rho^2\sigma^2\nu^2\tau(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)^2} \right)^{\frac{1}{2}}, \tag{I.B.5}$$

We can then proceed as in the Proof of Proposition 2 to show that $F(\theta^*, \mu)$ decreases with θ^*, which implies that $\theta^*(\mu)$ exists and is unique. Then, we can proceed as in the proofs of all other propositions and corollaries to show that these results still hold. The reason is that these results only depends on the direction of the effect of each exogenous parameters of the model (e.g., c, $\underline{\theta}$ or α) on $\bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)$ and $r(\theta, \theta^*, \mu)$, other things equal. The direction of this effect does not depend on μ. Last, holding θ^* constant, $r(\theta, \theta^*, \mu)$ increases with $\bar{\tau}(\theta^*; \mu, \underline{\theta}, \bar{\theta}, \alpha)$ and therefore μ. Thus, $F(\theta^*, \mu)$ increases with μ and therefore θ^* increases with μ (since in equilibrium $F(\theta^*, \mu) = \exp(-\rho c)$).

Moreover, in equilibrium $\bar{\tau}(\theta^*(\mu); \mu, \underline{\theta}, \bar{\theta}, \alpha)$ must increase with μ. Indeed, suppose that this is not the case (to be contradicted). Then, $F(\theta^*, \mu)$ would decrease with μ and therefore θ^* would decrease with μ. But then $\bar{\tau}(\theta^*(\mu); \mu, \underline{\theta}, \bar{\theta}, \alpha$ would increase with μ. A contradiction.

C. Supplement to the proof of Proposition 7

We prove that θ^* declines when c, or α, or $\underline{\theta}$ decreases but that in each of these cases, the average quality of speculators signals, $\bar{\tau}$ and therefore price informativeness increase. We refer to the equilibrium value of $\bar{\tau}$ as $\bar{\tau}^*$ for brevity (that is, $\bar{\tau}^* = \bar{\tau}(\theta^*(\mu^*); \mu^*, \underline{\theta}, \bar{\theta}, \alpha)$).

First, consider the effect of a decrease in c on $\theta^*(\mu^*)$. The equilibrium condition $F(\theta^*, \mu^*) = \exp(-\rho c)$ imposes that $F(\theta^*, \mu^*)$ decreases with c. Observe that $r(\theta, \theta^*, \mu^*)$ increases with $\bar{\tau}^*$ and decreases with θ^*. Thus, $F(\theta^*, \mu^*)$ also increases with $\bar{\tau}^*$ and decreases with θ^*. Now suppose to be contradicted that $\bar{\tau}^*$ increases with c. Then, for $F(\theta^*, \mu^*)$ to decrease, it must be that θ^* increases when c increases. Now remember that:

$$R(\mu^*, K) = \exp(\rho K) \left(\frac{\rho^2\sigma^2\nu^2\tau(\bar{\theta}) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*(\mu^*), \mu^*, \underline{\theta}, \bar{\theta})^2}{\rho^2\sigma^2\nu^2\tau(\theta^*(\mu^*)) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*(\mu^*), \mu^*, \underline{\theta}, \bar{\theta})^2} \right)^{\frac{1}{2}}. \tag{I.C.1}$$

Thus, $R(\mu^*, K)$ increases with θ^* and $\bar{\tau}^*$. It follows that if $\bar{\tau}^*$ and θ^* increase with c then $R(\mu^*, K)$ increases with c as well and the condition $R(\mu^*, K) = 1$ cannot be satisfied. A contradiction. It must therefore be that $\bar{\tau}^*$ and therefore price informativeness decreases with c. But then for the condition $R(\mu^*, K) = 1$, it must be that θ^* increases with c.

We deduce that Proposition 3 holds when speculators' entry is endogenous. It follows that all implications regarding c holds as well since they derive from Proposition 3. We can prove in the same way that the effect of α is as described in proposition 4 when speculators' entry is endogenous. It follows that all implications regarding α holds as well since they derive from Proposition 4.

Now consider the effect of a decrease in $\underline{\theta}$ on $\theta^*(\mu^*)$. The equilibrium condition $F(\theta^*, \mu^*) = \exp(-\rho c)$ imposes that $F(\theta^*, \mu^*)$ remains constant when $\underline{\theta}$ decreases. Remember that $r(\theta, \theta^*, \mu^*)$ increases with $\bar{\tau}^*$ and decreases with θ^*. Thus, $F(\theta^*, \mu^*)$ also increases with $\bar{\tau}^*$ and decreases with θ^*. Moreover, holding $\bar{\tau}^*$ constant, $F(\theta^*, \mu^*)$ increases with $\underline{\theta}$. Now suppose to be contradicted that $\bar{\tau}^*$ increases with $\underline{\theta}$. Then, for $F(\theta^*, \mu^*)$ to remain constant, it must be that θ^* increases when $\underline{\theta}$ increases. However, as explained previously $R(\mu^*, K)$ increases with θ^* and $\bar{\tau}^*$. It follows that if $\bar{\tau}^*$ and θ^* increase with $\underline{\theta}$ then $R(\mu^*, K)$ increases with $\underline{\theta}$ as well and the condition $R(\mu^*, K) = 1$ cannot be satisfied. A contradiction. It must therefore be that $\bar{\tau}^*$ and therefore price informativeness decrease with $\underline{\theta}$. But then for the condition $R(\mu^*, K) = 1$, it must be that θ^* increases with $\underline{\theta}$. Note that the latter condition is not required when the mass of speculators is exogenously fixed. This explains why in this case, θ^* can decrease when $\underline{\theta}$ increases while this is not possible when speculators' entry is endogenous.

D. Supplement to the proof of corollary 5

We refer to the equilibrium value of $\bar{\tau}$ as $\bar{\tau}^*$ for brevity (that is, $\bar{\tau}^* = \bar{\tau}(\theta^*(\mu^*); \mu^*, \underline{\theta}, \bar{\theta}, \alpha)$). Suppose to be contradicted that $\bar{\tau}^*$ increases with K. Now observe that $R(\mu^*, K)$ (see eq.(I.C.1)) increases with K, $\bar{\tau}^*$ and θ^*. Thus, for the equilibrium condition $R(\mu^*, K) = 1$ to be satisfied when K increases it must be that θ^* decreases with K if $\bar{\tau}^*$ increases with K. However, remember that in equilibrium, it must also be the case that $F(\theta^*, \mu^*) = \exp(-\rho c)$. Thus, when K increases, it must be the case that θ^* and $\bar{\tau}^*$ adjusts so that $F(\theta^*, \mu^*)$ remains constant. As $F(\theta^*, \mu^*)$ increases with $\bar{\tau}^*$ and decreases with θ^*, this implies that $\bar{\tau}^*$ must decrease with K when θ^* decreases with K. A contradiction.

Thus, when K increases, the average quality of speculators' signals, $\bar{\tau}^*$ and therefore price informativeness, decreases. It then follows from the fact that $F(\theta^*, \mu^*)$ must remain constant for all values of K that θ^* increases with K.

Part II

Extensions

A. Searching predictors with recall

In the baseline model, we assume that speculators's search for predictors is without recall: When they decide to stop searching, they must trade on the predictor they just discovered. An alternative assumption is that when they stop searching, speculators can use the best of all predictors found until they stop (i.e., they can "recall" predictors found in the past). In this section, we show that this extra flexibility does not affect the equilibrium stopping rule, θ^*.

Step 1. To do so, we first derive the continuation value of a speculator who follows an arbitrary stopping rule $\hat{\theta}$ when other speculators follow the stopping rule θ^*. A key difference with the baseline case is that this continuation value in a given exploration round depends on the best predictor she found until this round. Let denote this best predictor by θ^{min}. Observe that it must be the case that $\theta^{min} \geq \hat{\theta}$ (if the inequality was not satisfied the speculator would have stopped searching in a previous round; a contradiction). The expected utility of launching a new exploration for the speculator is:

$$J(\theta^{min}, \widehat{\theta}, \theta^*) = \exp(\rho c) \left[\alpha \left(\int_{\underline{\theta}}^{\widehat{\theta}} g(\theta', \theta^*) \phi(\theta') d\theta' + \int_{\widehat{\theta}}^{\theta^{min}} J(\theta', \widehat{\theta}, \theta^*) \phi(\theta') d\theta' \right) + \left(1 - \alpha \int_{\underline{\theta}}^{\theta^{min}} \phi(\theta') d\theta' \right) J(\theta^{min}, \widehat{\theta}, \theta^*) \right] \tag{II.A.1}$$

Differentiating both sides of eq.(II.A.1) with respect to θ^{min} implies that:

$$\frac{\partial J}{\partial \theta^{min}}\left(1-\exp(\rho c)\left(1-\alpha\int_{\underline{\theta}}^{\theta^{min}}\phi(\theta')d\theta'\right)\right)=\alpha\exp(\rho c)\left[J(\theta^{min},\widehat{\theta},\theta^{*})\phi(\theta)-\phi(\theta)J(\theta^{min},\widehat{\theta},\theta^{*})\right]=0 \tag{II.A.2}$$

Thus, $\frac{\partial J(\theta^{min},\widehat{\theta},\theta^{*})}{\partial\theta^{min}}=0$. Hence, $J(\theta^{min},\widehat{\theta},\theta^{*})$ does not depend on θ^{min} for $\theta^{min}>\widehat{\theta}$. This means that for $\theta^{min}>\theta'>\widehat{\theta}$, $J(\theta',\widehat{\theta},\theta^{*})=J(\theta^{min},\widehat{\theta},\theta^{*})$. Thus, eq.(II.A.1) implies:

$$J(\theta^{min},\widehat{\theta},\theta^{*})=J(\widehat{\theta},\theta^{*})=\exp(\rho c)\left[\alpha\int_{\underline{\theta}}^{\widehat{\theta}}g(\theta',\theta^{*})\phi(\theta')d\theta'+\left(1-\alpha\int_{\underline{\theta}}^{\widehat{\theta}}\phi(\theta')d\theta'\right)J(\widehat{\theta},\theta^{*})\right], \tag{II.A.3}$$

Hence, the continuation value of the speculator for any $\theta^{min}\geq\hat{\theta}$ is the same as in the baseline version of the model. Intuitively, the reason is that there is no limit to the number of explorations that a speculator can perform. Hence, the possibility for a speculator to "store" predictors has no value since the speculator keeps searching as long as the quality of the predictors she found is above $\widehat{\theta}$.

Step 2. Now, suppose that there exists a solution, $\widehat{\theta}$, to the following the indifference condition

$$g(\widehat{\theta},\theta^{*})=J(\widehat{\theta},\widehat{\theta},\theta^{*}). \tag{II.A.4}$$

We show below that $\widehat{\theta}$ is the optimal stopping rule of the speculator when other speculators' stopping rule θ^{*}. According to the one-shot deviation principle, a necessary and sufficient condition for $\widehat{\theta}$ to be optimal is that a one-shot deviation from this policy is not optimal. First, we show that it is always optimal to keep searching when $\theta>\widehat{\theta}$ (thus a one shot deviation that consists in stopping is not optimal), because

$$g(\theta,\theta^{*})<g(\widehat{\theta},\theta^{*})=J(\widehat{\theta},\widehat{\theta},\theta^{*})=J(\theta,\widehat{\theta},\theta^{*}), \tag{II.A.5}$$

where the first inequality follows from the fact that $g(\theta,\theta^{*})$ decreases with θ, the second equality follows from eq.(II.A.4), and the last equality follows from the fact that $J(\theta,\widehat{\theta},\theta^{*})$ does not depend on θ for $\theta\geq\widehat{\theta}$ (see Step 1).

Second, consider a case where the speculator's obtains a predictor of type $\theta\in[\underline{\theta},\widehat{\theta}]$. If stopping when $\theta<\widehat{\theta}$ is the optimal policy then a one shot deviation from this policy is not optimal. We now show that this is the case. The expected continuation value of the speculator with such a one shot deviation is:

$$X(\theta,\theta^*) = \exp(\rho c)\left[\alpha \int_{\underline{\theta}}^{\theta} g(\theta',\theta^*)\phi(\theta')d\theta' + \left(1-\alpha\int_{\underline{\theta}}^{\theta}\phi(\theta')d\theta'\right) g(\theta,\theta^*)\right]. \quad \text{(II.A.6)}$$

Indeed, either the speculator finds an even better predictor in the next exploration round (first term in the R.H.S of eq.(II.A.6)) or she does not and then her policy will command to stop at the next round with the best predictor found so far, i.e., the predictor of type θ (second term in eq.(II.A.6)). Notice that $X(\widehat{\theta},\theta^*) = J(\widehat{\theta},\widehat{\theta},\theta^*) = g(\widehat{\theta},\theta^*)$ which shows that a speculator with $\theta = \widehat{\theta}$ is indifferent between deviating or not. Now, we show that:

$$\forall\, \theta < \widehat{\theta},\ g(\theta,\theta^*) > X(\theta,\theta^*). \quad \text{(II.A.7)}$$

As $g(\theta,\theta^*) < 0$, using the expression for $X(\theta,\theta^*)$, this inequality is equivalent to

$$\forall\, \theta < \widehat{\theta},\ \exp(-\rho c) < Y(\theta,\theta^*), \quad \text{(II.A.8)}$$

where $Y(\theta,\theta^*) = \alpha\int_{\underline{\theta}}^{\theta}\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}\phi(\theta')d\theta' + 1 - \alpha\int_{\underline{\theta}}^{\theta}\phi(\theta')d\theta'$. Now:

$$\frac{\partial Y}{\partial \theta} = \int_{\underline{\theta}}^{\theta}\frac{\partial}{\partial\theta}\left(\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}\right)\phi(\theta')d\theta'. \quad \text{(II.A.9)}$$

The expected utility $g(\theta,\theta^*)$ is negative, and is decreasing in θ. Therefore $\frac{g(\theta',\theta^*)}{g(\theta,\theta^*)}$ is positive and also decreasing in θ. Thus, $\frac{\partial Y}{\partial\theta} < 0$ and therefore:

$$\forall\, \theta < \theta^*,\ \exp(-\rho c) < Y(\theta,\theta^*), \quad \text{(II.A.10)}$$

or equivalently $\forall\, \theta < \widehat{\theta},\ g(\theta,\theta^*) > X(\theta,\theta^*)$. Thus, $\widehat{\theta}$ is the optimal stopping rule for the speculator. Moreover, in a symmetric equilibrium, it must be the case that $\widehat{\theta}(\theta^*) = \theta^*$. This means that:

$$g(\theta^*,\theta^*) = J(\theta^*,\theta^*,\theta^*). \quad \text{(II.A.11)}$$

As the continuation value $J(\theta^*,\theta^*,\theta^*)$ is identical to that in the baseline model (see Step 1), the stopping rule θ^* is the same. Thus, the results are unchanged.

B. The distribution of predictors' quality

In this section, we first derive (in Step 1) the probability distribution of predictors' quality, τ as a function of the propbaility distribution of predictors' types, θ. Then, in Step 2, we show that the model in which speculators receive signals specified as $s_\theta = \cos(\theta)\omega + \sin(\theta)\varepsilon_\theta$ is identical to a model in which speculators receive signals specified as $\hat{s}_\tau = \omega + (\tau)^{-\frac{1}{2}}\varepsilon_\tau$. Thus one can use one or the other specifications. The latter is more standard but the former simplifies some calcualtions.

Step 1. Let $H(\tau(\theta))$ and $h(\tau(\theta))$ be respectively the cumulative probability distribution of τ and the density of τ. By definition $\tau(\theta) = \cot^2(\theta) = 1/\sin^2(\theta) - 1$. Thus, $\theta(\tau) = \arcsin[(1+\tau)^{-1/2}]$. We deduce that

$$H(\tau) = 1 - \Phi\left(\arcsin[(1+\tau)^{-1/2}]\right). \tag{II.B.1}$$

Moreover, as $\frac{\partial \arcsin(x)}{\partial x} = (1-x)^{-1/2}$, we deduce that:

$$h(\tau) = \frac{1}{2(1+\tau)\tau^{1/2}}\phi(\arcsin[(1+\tau)^{-1/2}]) \tag{II.B.2}$$

Step 2. Now suppose that we consider a model in which predictors are expressed as follows: $\hat{s}_\tau = \omega + (\tau)^{-\frac{1}{2}}\varepsilon_\tau$ and τ is distributed according to $h(\tau)$ on $[0,\infty)$. In this specification, a predictor is directly characterized by its precision, τ. The informativeness of a predictor of type τ is identical to the informativeness of a predictor $s_{\theta(\tau)} = \cos(\theta(\tau))\omega + \sin(\theta(\tau))\varepsilon_\theta$, such that $\theta(\tau) = \arcsin[(1+\tau)^{-1/2}]$. Indeed, we have:

$$\mathsf{Var}(\omega \mid \hat{s}_\tau) = \mathsf{Var}(\omega \mid s_{\theta(\tau)}) = \frac{\sigma_\omega^2}{1+\tau} \tag{II.B.3}$$

Thus, speculators' decisions are identical when they can search predictors such that $\hat{s}_\tau = \omega + (\tau)^{-\frac{1}{2}}\varepsilon_\tau$ or predictors such that $s_\theta = \cos(\theta)\omega + \sin(\theta)\varepsilon_\theta$, provided that the distribution of θ ($\phi(.)$) and τ ($h(.)$) are consistent with each other, i.e., satisfy eq.(II.B.2)). This implies that our results hold when predictors are specified as $\hat{s}_\tau = \omega + (\tau)^{-\frac{1}{2}}\varepsilon_\tau$.

C. Searching predictors by combining signals

In this section, we propose one formulation of the process by which speculators build their predictors. This formulation makes clear why the quality of a speculator's predictor in a given round is not necessarily greater than in a previous round, as assumed in our baseline model. To see this, suppose that in a given round speculators can use N variables (signals) s_j to predict the asset payoff ω. We formalize each variable as being a signal about the asset payoff. Specifically, we assume that

$$s_j = \omega + (\tau_j)^{-\frac{1}{2}}\epsilon_j, \quad j \in \{1, ..., N\} \tag{II.C.1}$$

where the ϵ_js' have a normal distribution with mean zero and precision $\tau_\omega = 1/\sigma^2$. Importantly, we assume that τ_j is specific to each variable used in a given round and can therefore vary across variables. Moreover, we assume that τ_j is drawn from some distribution. The key assumption is that the number of variables used to predict the asset payoff in a given exploration round is fixed (N can be large but it cannot increase as new rounds are launched). This means that in a given round, a speculator must replace at least one of the variable used in the past by a new variable whose predictive quality (τ_j) is unknown (it is discovered in the exploration round). In reality, research teams may fix the number of variables used in their predictive models to avoid overfitting. Thus, we think that the assumption that N is fixed (it can be large) is reasonable. As explained below, this assumption implies that the quality of a predictor in a given round is not necessarily larger as in previous rounds (as what would happen if one could retain variables from past explorations and add new ones). Moreover, we show below that the quality τ of a predictor in a given round is the sum of the quality of each variable used to form the predictor (i.e., $\tau = \sum_{j=1}^{j=N} \tau_j$).

To see this formally, fix the variables used in the n^{th} round of exploration and denote by $\tau_j(n)$ the realization of the precision for signal s_j in round n. Similarly, let $\tau(n) = \sum_{j=1}^{j=N} \tau_j(n)$ be the sum of these precisions in the n^{th} round. Using standard properties of normally distributed variables, we deduce that:

$$\mathsf{E}(\omega \mid s_1(n), s_2(n), ..., s_N(n)) = \sum_{j=1}^{j=N} \mu_j(n) s_j, \tag{II.C.2}$$

with $\mu_j(n) = \frac{\tau_j(n)}{1+\tau(n)}$. As all variables are normally distributed, $\mu_j(n)$ is the coefficient that would be obtained for the j^{th} variable in running a regressing of ω on the N variables used in round n. The predictor obtained in this round is just the predicted value of this regression, i.e., $\mathsf{E}(\omega \mid s_1(n), s_2(n), ..., s_N(n))$. Thus, the predictor obtained in round n is:

$$s_{\tau(n)} = \sum_{j=1}^{j=N} \mu_j(n) s_j = \left(\sum_j \mu_j(n) \right) \omega + \sum_j \mu_j(n)(\tau_j(n))^{-\frac{1}{2}} \epsilon_j. \tag{II.C.3}$$

Alternatively, one can use as predictor (its informativeness is identical):

$$\hat{s}_{\tau(n)} = \omega + \sum_j \frac{\mu_j(n)}{\sum_j \mu_j(n)} (\tau_j(n))^{-\frac{1}{2}} \epsilon_j. \tag{II.C.4}$$

Using the definition of $\mu_j(n)$, this can be rewritten:

$$\hat{s}_{\tau(n)} = \omega + (\tau(n))^{-\frac{1}{2}} \epsilon_{\tau(n)}. \tag{II.C.5}$$

where $\epsilon_{\tau(n)} = \sum_j \frac{(\tau_j(n))^{\frac{1}{2}}}{\left(\sum_j \tau_j(n)\right)^{\frac{1}{2}}} \epsilon_j$. Thus, this process generates predictors that have exactly the structure of those considered in our model (Section II.B in this appendix shows that this specification of predictors is equivalent to that considered in the baseline model). Thus, $\tau(n)$ is the quality of the predictor built with N signals in the n^{th} round. Note that the theoretical R^2 of the regression of ω on the N variables used in round n (i.e., $1 - \mathsf{Var}[\omega \mid \hat{s}_{\tau(n)}] / \mathsf{Var}[\omega]$) is equal to $\tau(n)(1+\tau(n))^{-1}$. Thus, the higher the quality of a predictor, the higher the R^2 of a regression of the asset payoff on the predictor.

Now suppose that a speculator decides to launch a $(n+1)^{th}$ round of exploration by considering a different set of variables. As at least one variable is different, $\tau(n+1)$ for the predictor built with these variables will be different from $\tau(n)$. Moreover as the precision of the signals used in one round are random, the precision of the predictor obtained in the $(n+1)^{th}$ round can be larger or smaller than that of the previous predictor. For instance, suppose that in each round of exploration, τ_j follows a Gamma distribution with parameters γ_0 and γ_1. Then, if the speculator changes all the variables used in the $(n+1)^{th}$ round of exploration, the probability that $\tau(n+1)$ is smaller than $\tau(n)$ is $G(\tau(n); N\gamma_0, \gamma_1)$ where $G(.)$ is the cumulative probability distribution of a Gamma distribution with parameters $N\gamma_0$ and γ_1. Clearly, the same possibility arises even if the speculator only considers changing a subset of all variables used in the previous round.

D. The likelihood of finding a predictor: Alternative Specification

In the model, we assume that in a given round, speculators draw the type of their predictors according to the unconditional distribution of predictors' type ($\phi(.)$) in the interval $[0, \pi/2]$ but that they cannot exploit predictors with a type $\theta < \underline{\theta}$. Alternatively, we could assume that speculators draw the type of their predictors in $[\underline{\theta}, \pi/2]$, conditional on this type being in this interval. In this section, we show that this approach yields the same findings and in fact enlarges the set of values for $\underline{\theta}$ for which a decrease in $\underline{\theta}$ results in an increase in θ^*.

With this approach, the likelihood that a speculator finds a predictor of type θ is:

$$\psi(\theta, \underline{\theta}) = \frac{\phi(\theta)}{1 - \Phi(\underline{\theta})}. \tag{II.D.1}$$

Thus, if a speculator uses the stopping rule θ_i^*, she finds a predictor in a given round with probability:

$$\Lambda^{new}(\theta_i^*; \underline{\theta}, \alpha) \equiv \alpha \Pr(\theta \in [\underline{\theta}, \theta_i^*] \mid \theta \in [\underline{\theta}, \frac{\pi}{2}]) = \alpha \times \frac{\Phi(\theta_i^*) - \Phi(\underline{\theta})}{1 - \Phi(\underline{\theta})}. \tag{II.D.2}$$

We can then proceed as in Section 4 in the paper to write the continuation value $J(\widehat{\theta}_i, \theta^*)$ of speculator i after she turns down a predictor, replacing $\Lambda(\theta_i^*; \underline{\theta}, \alpha)$ in (eq.(14) in the text by $\Lambda^{new}(\theta_i^*; \underline{\theta}, \alpha)$. Doing so and simplifying yields

$$J(\widehat{\theta}_i, \theta^*) = \exp(\rho c) \left(\frac{\alpha}{1 - \Phi(\underline{\theta})} \mathsf{E}\left[g(\theta, \theta^*) \mid \underline{\theta} \leq \theta \leq \widehat{\theta}_i \right] + (1 - \Lambda^{new}(\widehat{\theta}_i; \underline{\theta}, \alpha)) J(\widehat{\theta}_i, \theta^*) \right). \tag{II.D.3}$$

We can then proceed as in the text to obtain the equilibrium condition:

$$F(\theta^*) = exp(-\rho c),$$

where

$$F(\theta^*) = \alpha \frac{\int_{\underline{\theta}}^{\theta^*} r(\theta, \theta^*) \phi(\theta) d\theta}{1 - \Phi(\underline{\theta})} + (1 - \Lambda^{new}(\widehat{\theta}_i; \underline{\theta}, \alpha)). \tag{II.D.4}$$

On can then follow the same steps as in the text to show that Propositions 3 and 4 still hold in this case. In particular, consider the effect of $\underline{\theta}$. Taking the derivative of $F(\dot{})$ in eq.(II.D.4) with respect to $\underline{\theta}$, one obtains

$$\begin{aligned}\frac{\partial F}{\partial \underline{\theta}} = & \frac{\alpha}{1-\Phi(\underline{\theta})}\left((1-r(\underline{\theta},\theta^*))\phi(\underline{\theta}) + \int_{\underline{\theta}}^{\theta^*}\frac{\partial r}{\partial \underline{\theta}}\phi(\theta)d\theta\right) \\ & \underbrace{-\alpha\frac{\phi(\underline{\theta})}{\left(1-\Phi(\underline{\theta})\right)^2}\int_{\underline{\theta}}^{\theta^*}(1-r(\theta,\theta^*))\phi(\theta)d\theta}_{<0}\,. \end{aligned} \tag{II.D.5}$$

For θ^* to increase when $\underline{\theta}$ decreases, it must be that $\frac{\partial F}{\partial \underline{\theta}} < 0$. In the case considered in the baseline version of our model, the last term in eq.(II.D.5) is not present and we have established (in the proof of Proposition 4) that for $\underline{\theta} < \underline{\theta}^{tr}$, the first term in parenthesis in eq.(II.D.5) is negative. This condition is clearly sufficient here as well since the last term in eq.(II.D.5) is negative. Moreover, precisely because this term is negative, the range of value for $\underline{\theta}$ for which $\frac{\partial F}{\partial \underline{\theta}} < 0$ is wider than in the baseline case. Thus, the threshold value for $\underline{\theta}$ such that $\frac{\partial F}{\partial \underline{\theta}} < 0$ is larger than in the baseline model when we use the approach outlined in this section.

Part III

A specific distribution for predictors' types

In this section, we present a family of probability distributions for $\phi(.)$ for which the function $F(.)$ can be computed in closed-form (Section III).A) and we derive the corresponding probability distribution for the predictors' quality τ. As $F(.)$ can be computed in closed-form, the equilibrium stopping rule (θ^*) as well as various variables of interest that depends on the equilibrium stopping rule (e.g., the mean and the variance of the distribution of the quality of predictors across speculators) can be computed as well (at least numerically). In Section III.B, we focus on the two special cases of this family of distributions that we use in the numerical examples considered in the paper.

A. A family of distribution for predictors' types

Consider the following family of distribution for predictors' types θ, indexed by $n \geq 1$ such that:

$$\Phi(\theta) = \sin^{2n+1}(\theta), \quad \text{and therefore} \quad \phi(\theta) = (2n+1)\cos(\theta)\sin^{2n}(\theta). \tag{III.A.1}$$

For this family of distribution, the cumulative probability distribution, $H(.)$, of $\tau(\theta)$ and its density $h(.)$ are readily derived using eq.(II.B.1) and eq.(II.B.2) in Section II.B of this appendix. We obtain:

$$H(\tau) = 1 - \frac{1}{(1+\tau)^{n+\frac{1}{2}}}. \tag{III.A.2}$$

and

$$h(\tau) = \frac{n+\frac{1}{2}}{(1+\tau)^{n+\frac{3}{2}}}. \tag{III.A.3}$$

Thus, $(1+\tau)$ has a power law distribution with parameters $(1, n+3/2)$.

Moreover, for this family of probability distribution, one can easily compute $\bar{\tau}(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\tau(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]$. Indeed, as $\tau(\theta) = \cot(\theta)^2$, we have:

$$\begin{aligned}
\mathsf{E}\left[\tau(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right] &= \int_{\underline{\theta}}^{\theta^*} \frac{\cos^2(\theta)}{\sin^2(\theta)}(2n+1)\cos(\theta)\sin^{2n}(\theta)d\theta \\
&= \int_{\underline{\theta}}^{\theta^*} (2n+1)\cos^3(\theta)\sin^{2n-2}(\theta)d\theta \\
&= \int_{\underline{\theta}}^{\theta^*} (2n+1)(\cos(\theta)\sin^{2n-2}(\theta) - \cos(\theta)\sin^{2n}(\theta))d\theta \\
&= \left[\frac{2n+1}{2n-1}\sin^{2n-1}(\theta) - \sin^{2n+1}(\theta)\right]_{\underline{\theta}}^{\theta^*}
\end{aligned} \tag{III.A.4}$$

Remember that the equilibrium stopping rule θ^* solves: $\exp(-\rho c) = F(\theta^*)$ where ((see eq.(19) in the text):

$$F(\theta^*) \equiv \alpha \int_{\underline{\theta}}^{\theta^*} r(\theta, \theta^*)\phi(\theta)d\theta + (1 - \Lambda(\theta^*; \underline{\theta}, \alpha)), \quad \text{for } \theta^* \in \left[\underline{\theta}, \frac{\pi}{2}\right], \tag{III.A.5}$$

with

$$r(\theta, \theta^*) = \frac{g(\theta, \theta^*)}{g(\theta^*, \theta^*)} = \left(\frac{\rho^2\sigma^2\nu^2\tau(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)}{\rho^2\sigma^2\nu^2\tau(\theta) + \rho^2\sigma^2\nu^2 + \bar{\tau}^2(\theta^*; \underline{\theta}, \alpha)}\right)^{\frac{1}{2}}, \tag{III.A.6}$$

where the expression for $r(\theta,\theta^*)$ follows from eq.(52) in the appendix of the paper.

Thus, to compute $F(.)$, we just needs to compute $\bar{\tau}(\theta^*;\underline{\theta},\alpha)$ and the integral in eq.(III.A.5). The expression for $\bar{\tau}(\theta^*;\underline{\theta},\alpha)$ is given by eq.(III.A.4). Thus, we just need to explain how to compute the following integral.

$$\int_{\underline{\theta}}^{\theta^*} \frac{\phi(\theta)}{(\rho^2\sigma^2\nu^2\cot^2(\theta)+\rho^2\sigma^2\nu^2+\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)^2)^{\frac{1}{2}}}d\theta = (2n+1)D(0,n) \tag{III.A.7}$$

where

$$D(0,n) \equiv \int_{\underline{\theta}}^{\theta^*} \cos(\theta)\sin^{2n+1}(\theta)\left(\rho^2\sigma^2\nu^2+\bar{\tau}^2(\theta^*;\underline{\theta},\alpha)^2\sin^2(\theta)\right)^{-\frac{1}{2}}d\theta \tag{III.A.8}$$

We now explain how to compute $D(0,n)$. To this end, let define $D(k,m)$ as

$$D(k,m) \equiv \int_{\underline{\theta}}^{\theta^*} \cos(\theta)\sin^{2m+1}(\theta)\left(\rho^2\sigma^2\nu^2+\bar{\tau}(\theta^*;\underline{\theta},\alpha)^2\sin^2(\theta)\right)^{k-\frac{1}{2}}d\theta.$$

Integrating by part, one obtains

$$\begin{aligned} D(k,m) = &\left[\frac{1}{2\left(k+1-\frac{1}{2}\right)}\sin^{2m}(\theta)\left(\rho^2\sigma^2\nu^2\tau(\theta)+\rho^2\sigma^2\nu^2+\bar{\tau}(\theta^*;\underline{\theta},\alpha)^2\right)^{k+1-\frac{1}{2}}\right]_{\underline{\theta}}^{\theta^*} \\ &-\frac{2m}{2\left(k+1-\frac{1}{2}\right)}D(k+1,m-1) \end{aligned} \tag{III.A.9}$$

and

$$D(k,0) = \left[\frac{1}{2\left(k+1-\frac{1}{2}\right)}\left(\rho^2\sigma^2\nu^2\tau(\theta)+\rho^2\sigma^2\nu^2+\bar{\tau}(\theta^*;\underline{\theta},\alpha)^2\right)^{k+1-\frac{1}{2}}\right]_{\underline{\theta}}^{\theta^*}$$

Hence, $D(0,n)$ can be expressed as a function of $D(1,n-1)$, which can be expressed as a function of $D(2,n-2)$ etc. until $D(n,0)$. Thus, one can obtain a closed-form expression for $D(0,n)$ and therefore $F(.)$ for any n.

B. Special cases: n=1 and n=2

The numerical examples used in the paper correspond to the cases $n=1$ and $n=2$ for the family of distributions presented in the previous section. Figures III.B.1 and III.B.2

below plot the density of θ in these two particular cases.

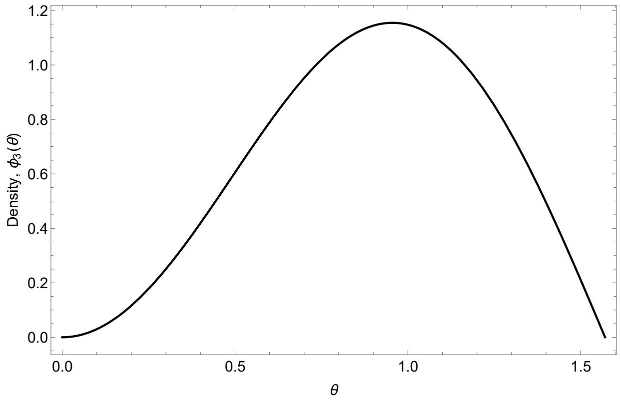

Figure III.B.1: This graph represents the distribution function $\phi_(\theta) = 3\cos(\theta)\sin^2(\theta)$ on the interval $[0, \pi/2]$.

and

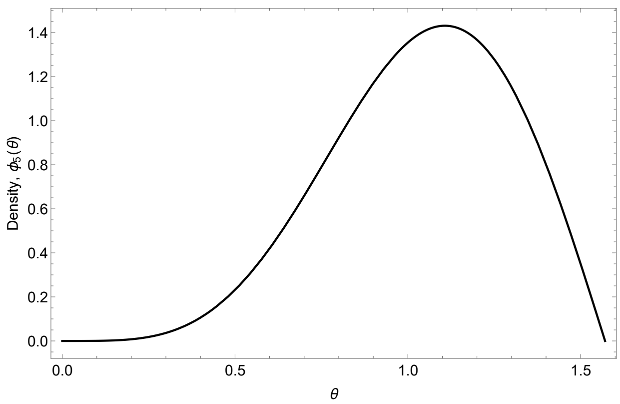

Figure III.B.2: This graph represents the distribution function $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$ on the interval $[0, \pi/2]$.

In these two cases, we derive in Lemma 1 and 2 closed form expressions for the two first moments of the distribution of the quality of predictor chosen by speculators in equilibrium, i.e., $\bar{\tau}(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\tau(\theta')| \, \underline{\theta} \leq \theta' \leq \theta^*\right]$ and $m_2(\theta^*, \underline{\theta}) \equiv \mathsf{E}\left[\tau^2(\theta')| \, \underline{\theta} \leq \theta' \leq \theta^*\right]$ and $F(\theta^*)$.

B.1 Case 1: $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$ ($n = 1$).

Lemma 1. *When $\phi(\theta) = 3\cos(\theta)\sin^2(\theta)$, we have:*

$$\bar{\tau}(\theta^*, \underline{\theta}, \alpha) = \frac{3[\sin(\theta^*) - \sin(\underline{\theta})] - [\sin^3(\theta^*) - \sin^3(\underline{\theta})]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \tag{III.B.1}$$

$$m_2(\theta^*, \underline{\theta}) = \frac{\left[3\left(\frac{1}{\sin(\underline{\theta})} - \frac{1}{\sin(\theta^*)}\right) - 6(\sin(\theta^*) - \sin(\underline{\theta})) + \sin^3(\theta^*) - \sin^3(\underline{\theta})\right]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \quad \text{(III.B.2)}$$

and

$$F(\theta^*) = 1 - (\sin^3(\theta^*) - \sin^3(\underline{\theta})) \quad \text{(III.B.3)}$$
$$+ \left(\rho^2\sigma^2\nu^2 \cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2\right)^{\frac{1}{2}} [\delta(\theta^*) - \delta(\underline{\theta})] \quad \text{(III.B.4)}$$

where $\delta(\theta) = \frac{\left(\rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^2 \sin^2(\theta)\right)^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^4} (\bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2 \sin^2(\theta) - 2\rho^2\sigma^2\nu^2)$.

This result, combined with the expressions for the first and the second moments in Lemma 1 enable us to compute, for instance, the first and the second moment of speculators' expected trading profit in Section 5.3. ($\mathsf{E}[\bar{\pi}(\theta)]$ and $\mathsf{Var}[\bar{\pi}(\theta)]$).

Proof of Lemma 1. The expressions for $\bar{\tau}(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\tau(\theta') | \, \underline{\theta} \leq \theta' \leq \theta^*\right]$ and $F(\theta^*)$ follows from the derivations in Section III.A for the general case. Moreover:

$$m_2(\theta^*, \underline{\theta}, \alpha) \equiv \mathsf{E}\left[\cot^4(\theta') \middle| \, \underline{\theta} \leq \theta' \leq \theta^*\right] \quad \text{(III.B.5)}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 3\frac{\cos^5(\theta)}{\sin^2(\theta)} d\theta \quad \text{(III.B.6)}$$
$$= \frac{1}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \int_{\underline{\theta}}^{\theta^*} 3\cos(\theta) \left(\frac{1}{\sin^2(\theta)} - 2 + \sin^2(\theta)\right) d\theta \quad \text{(III.B.7)}$$
$$= \frac{\left[3\left(\frac{1}{\sin(\underline{\theta})} - \frac{1}{\sin(\theta^*)}\right) - 6(\sin(\theta^*) - \sin(\underline{\theta})) + \sin^3(\theta^*) - \sin^3(\underline{\theta})\right]}{\sin^3(\theta^*) - \sin^3(\underline{\theta})} \quad \text{(III.B.8)}$$

B.2 Case 2: $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$ $(n = 2)$.

Lemma 2. *When* $\phi(\theta) = 5\cos(\theta)\sin^4(\theta)$*,*

$$\bar{\tau}(\theta^*, \underline{\theta}, \alpha) = \frac{\frac{5}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] - [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})}. \quad \text{(III.B.9)}$$

$$m_2(\theta^*, \underline{\theta}) = \frac{5[\sin(\theta^*) - \sin(\underline{\theta})] - \frac{10}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] + [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})} \quad \text{(III.B.10)}$$

and

$$F(\theta^*) = 1 - (\sin^5(\theta^*) - \sin^5(\underline{\theta})) \quad \text{(III.B.11)}$$
$$+ \left(\rho^2\sigma^2\nu^2 \cot^2(\theta^*) + \rho^2\sigma^2\nu^2 + \bar{\tau}(\theta^*, \underline{\theta}, \alpha)^2\right)^{\frac{1}{2}} [\delta(\theta^*) - \delta(\underline{\theta})] \quad \text{(III.B.12)}$$

with $\delta(\theta) = \frac{\left(\rho^2\sigma^2\nu^2+\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta)\right)^{\frac{1}{2}}}{\bar{\tau}(\theta^*,\underline{\theta},\alpha)^6}\Big[\frac{8}{3}\rho^4\sigma^4\nu^4 - \frac{4}{3}\rho^2\sigma^2\nu^2\bar{\tau}(\theta^*,\underline{\theta},\alpha)^2\sin^2(\theta) + \bar{\tau}(\theta^*,\underline{\theta},\alpha)^4\sin^4(\theta)\Big].$

As in case 1, the expression for the second moment allows us to compute the volatility of speculators expected profits, $\mathsf{Var}[\pi(\theta)]$.

Proof of Lemma 2. The expressions for $\bar{\tau}(\theta^*,\underline{\theta},\alpha) \equiv \mathsf{E}\left[\tau(\theta')\middle|\,\underline{\theta} \leq \theta' \leq \theta^*\right]$ and $F(\theta^*)$ follows from the derivations in Section III.A for the general case. Moreover:

$$\begin{aligned}
m_2(\theta^*,\underline{\theta}) &\equiv \mathsf{E}\left[\cot^4(\theta')\middle|\,\underline{\theta} \leq \theta' \leq \theta^*\right] \\
&= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 5\cos^5(\theta)d\theta \\
&= \frac{1}{\sin^5(\theta^*) - \sin^5(\underline{\theta})}\int_{\underline{\theta}}^{\theta^*} 5\cos(\theta)(1-2\sin^2(\theta)+\sin^4(\theta))d\theta \\
&= \frac{5[\sin(\theta^*) - \sin(\underline{\theta})] - \frac{10}{3}[\sin^3(\theta^*) - \sin^3(\underline{\theta})] + [\sin^5(\theta^*) - \sin^5(\underline{\theta})]}{\sin^5(\theta^*) - \sin^5(\underline{\theta})}
\end{aligned} \tag{III.B.13}$$